Distinguishing Maximally Entangled States by PPT Operations and Entanglement Catalysis Discrimination

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In order to better understand the class of quantum operations that preserve the positivity of partial transpose (PPT operations) and its relation to the widely used class of local operations and classical communication (LOCC), we study the problem of distinguishing orthogonal maximally entangled states (MES) by PPT operations. Firstly, we outline a rather simple proof to show that the number of $d\otimes d$ PPT distinguishable MES is at most d, which slightly generalizes existing results on this problem. Secondly, we construct 4 MES in $4\otimes 4$ state space that cannot be distinguished by PPT operations. Before our work, it was unknown whether there exists d MES in $d\otimes d$ state space that are locally indistinguishable. This example leads us to a novel phenomenon of "Entanglement Catalysis Discrimination". Moreover, we find there exists a set of locally indistinguishable states K such that $K^{\otimes m}$ is locally distinguishable for some finite m. As an interesting application, we exhibit a quantum channel with one sender and two receivers, whose one-shot zero-error local capacity is not optimal, but multi-use would enhance the capacity to achieve the full output dimension even without entangled input. Finally, we consider the entanglement cost of distinguishing three Bell states and a $2\otimes 2$ entangled basis. In the former case a bipartite pure entangled state with the largest Schmidt coefficient at most 2/3 is necessary and sufficient, while in the latter case an additional Bell state, or one ebit, should be supplied.

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I. INTRODUCTION

One of the most striking features of quantum mechanics is that quantum composite system can exhibit nonlocality. This can witnessed by the fact that there exist some quantum operations acting on bipartite systems cannot be implemented by local operations and classical communication (LOCC) only. Indeed identifying whether various information tasks can be accomplished by LOCC is becoming a very general strategy for studying quantum nonlocality. Although many fascinating results have been obtained, there is still severe obstacle for understanding the structure of LOCC operations.

Among these tasks, local discrimination of orthogonal states is an effective one and has recently attracted quite considerable attention, see Refs. [1–10] and references therein for a number of such progresses. Two extreme cases, namely the discrimination of product orthogonal states and the discrimination of maximally entangled states (MES), are very of special interest. Lots of effort has been devoted to the discrimination of orthogonal product states [11–14]. Most notably, in Ref.[11], Bennett et al discovered a $3\otimes 3$ orthonormal product basis that are indistinguishable by LOCC. Shortly after that it was further shown in Ref. [12] that the members of an orthogonal unextendible product basis(UPB) are not perfectly distinguishable by LOCC, where it was also shown that any set of orthogonal product states on $2\otimes n$ is locally distinguishable. In Ref. [14], a complete characterization of locally indistinguishable product states on $2\otimes 2\otimes 2$ and $3\otimes 3$ was presented. On the other hand, local discrimination of MES was studied in Refs. [1–4, 15, 16]. In particular, in Ref. [3] it was shown that any three Bell states,i.e., $2\otimes 2$ MES, are not distinguished locally. Later it was proved that in $d\otimes d$ system any $k > d$ MES are not exactly distinguishable by separable operations in [1, 2, 15, 16]. At the same time, in Ref. [1] Nathanson also showed that any three $3\otimes 3$ MESs can be distinguished by local projective measurement and one-way classical communication [1]. Unfortunately, it was unknown whether

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$d+1$ is always a tight bound for the number of locally distinguishable MES. This issue was discussed in Ref. [4] but no affirmative answer was known.

In this paper we study the discrimination of quantum states by operations that completely preserve the positivity of partial transpose (PPT operations for short). The significance of PPT operations can be highlighted as follows. First, the mathematical structure of PPT operations are rather simple than LOCC and Separable operations, and in many cases it provides a good approximation to LOCC. It is well known that LOCC is a proper subset of PPT operations. The positive operator-valued measures (POVMs)that can be implemented by operations in these two different classes can be characterized as follows: An LOCC POVM is the one that can be implemented as a multi-round process where in each round a local measurement based on previous measurement outcomes is preformed on one party, and then the outcome is broadcast; A PPT POVM, on the other hand, is simply a POVM with each measure operator having positive partial transpose. More precisely, a POVM $(M_i)_{i=0}^{n-1}$ acting on a bipartite system is said to be PPT if and only if $M_i \geq 0$ and $M_i^{\Gamma_A} \geq 0$ hold for any i with $\sum_i M_i = I$, where Γ_A means the partial transpose according to part A, i.e., $(|ij\rangle\langle kl|)^{\Gamma_A} = |kj\rangle\langle il|$. (For simplicity, in the following discussions we will omit the subscript "A"). From their descriptions we can clearly see that LOCC operations are rather complicated than PPT operations, and the later actually has very simple and feasible structure. Many tasks using PPT operations can be solved efficiently by employing semi-definite programming. For instance, this technique has been used to study the problem of entanglement distillation and pure state transformations under PPT operation in Ref [17] and Ref. [18], respectively. PPT operations are also extremely useful when we want to prove that some tasks cannot be achieved by LOCC.

Second, PPT operations play a significant role in entanglement theory. A celebrated result due to Horodecki et al establishes a connection between separability and positive maps acting on operators [19]. They used this connection to prove that the PPT criterion for separability introduced by Peres [20] is a necessary and sufficient condition for separability of a $2\otimes 2$ or $2\otimes 3$ state. Latter, in Ref.[21], it was further shown that if a mixed state can be distilled to the singlet form, it must violate the PPT criterion. This result classifies entanglement into two disjoint classes: "free" entanglement which is distillable, and "bound" entanglement which cannot be boosted into the singlet form by LOCC. It has been conjectured NPPT bound entangled does exist, and this remains one of the most important open problems in quantum information theory. Further exploring the properties in quantum information processing may be helpful in resolving this conjecture.

PPT discrimination of quantum states seems almost untouched in the previous literatures. We attempt to provide some preliminary results on this topic by studying the PPT distinguishability of MES. Our main results are summarized as follows. In Sec. II we provide a rather simple proof to show that any $k > d$ MES over $d\otimes d$ state space are PPT indistinguishable. This result seems slightly stronger than the previous results in Ref. [1, 16] where only separable operations were employed. However, we find that this result was implicitly implied in Ref. [2]. We include such a proof as we believe it may be helpful to certain reader.

In Sec. III we explicitly construct a set S consisting of four $4\otimes 4$ MES that are indistinguishable by PPT (thus also LOCC indistinguishable). To the best of our knowledge, our result is the first example of d MES over $d\times d$ state space that are locally indistinguishable. Due to the special structure of our example, we further observe a quite surprising "Entanglement Catalyst Discrimination" phenomenon happening on S. More precisely, with a $2\otimes 2$ MES as resource, we can locally distinguish among the members of S, and after the discrimination, we are still left with another two-qubit MES.

In Sec. IV, we show that tensor product could increase the local distinguishability sometimes. More precisely, we construct a set of states K (based the set of states S above) with the following property: K is locally indistinguishable while $K^{\otimes m}$ becomes locally distinguishable for some finite m. Such set K has an interesting application in the zero-error local classical capacity of quantum channels [22]. Actually it enables us to construct a $c-q$ (classical input and quantum output) channel with one sender and two receivers such that the one-shot zero-error local capacity is not optimal, i.e. strictly smaller the dimension of the output space, but multi-shot can achieve the full dimension of the output space. Note such channel does not exist in the global setting [23].

Finally in Sec. V we employ similar techniques to compute the entanglement cost of distinguishing three Bell states and two-qubit basis by PPT operations. Remarkably, we show to distinguish any three Bell states with PPT operations, a bipartite pure entangled state with the largest Schmidt coefficient at most $2/3$ is necessary and sufficient. In contrast, any $2\otimes 2$ basis containing one MES requires one $2\otimes 2$ MES to achieve exact discrimination. Some proof details of our main results are presented in Appendices A,B, and C.

II. INDISTINGUISHABILITY OF $d\otimes d$ MES

In this section, we will derive an upper bound on the number of PPT distinguishable MES.

Theorem 1. *No $k > d$ MES in $d\otimes d$ state space can be perfectly distinguished under PPT POVM.*

Proof:—We shall first show that if E satisfies $E, E^\Gamma \geq 0$ and $E|\Phi\rangle = |\Phi\rangle$, then $\mathrm{Tr}E \geq d$, where $|\Phi\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|jj\rangle$. To see this, let

$$G = \int_V (V \otimes V^*)E(V \otimes V^*)^\dagger dV,$$

where V ranges over all unitaries.

It is easy to see that there is some $a, b \in \mathcal{R}$ such that

$$G = aI_{d^2} + b|\Phi\rangle\langle\Phi|, \quad \text{then} \quad G^\Gamma = aI_{d^2} + bS/d \geq \ 0,$$

where I_{d^2} is identity gate and S is swap gate on $d \otimes d$ state space.

Then

$$G|\Phi\rangle = |\Phi\rangle, G^\Gamma \geq 0 \Rightarrow \ a + b = 1, \ a - b/d \geq 0 \ \Rightarrow \ a \geq 1/(d+1).$$

Now $\mathrm{Tr}E = \mathrm{Tr}G = ad^2 + b = 1 + (d^2-1)a \geq d$.

Suppose $\{(I \otimes U_i)|\Phi\rangle : 0 \leq i \leq k-1\}$ can be distinguished by PPT POVM $(E_i)_{i=0}^{k-1}$, where U_i are unitaries. We will have that $F_i|\Phi\rangle = |\Phi\rangle$, and $F_i, F_i^\Gamma \geq 0$, where $F_i = (I \otimes U_i)^\dagger E_i (I \otimes U_i)$. Thus, $\mathrm{Tr}E_i = \mathrm{Tr}F_i \geq d$.

$$d^2 = \mathrm{Tr}I_{d^2} = \mathrm{Tr}(\sum_{i=0}^{k-1} E_i) = \sum_{i=0}^{k-1}\mathrm{Tr}E_i \geq kd \Rightarrow k \leq d.$$

Then the conclusion holds. ■

We should point out that the above theorem can be obtained by combining with the methods in Refs.[2] and [24] since the proof of "Robustness of bipartite pure states" also implies $\mathrm{Tr}E \geq d$ under the conditions $I \geq E \geq 0$, $E^\Gamma \geq 0$ and $E|\Phi\rangle = |\Phi\rangle$. We hope that the above proof may be helpful as it is rather short and conceptually simple.

III. PPT INDISTINGUISHABLE MES AND ENTANGLEMENT CATALYSIS DISCRIMINATION

Now it would be desirable to know whether there is any locally indistinguishable set consisting of d MES in $d \otimes d$ system. We will settle this problem by presenting a PPT indistinguishable example in case of $d = 4$. More precisely, we show that $S = \{|\chi_i\rangle_{AB} : 0 \leq i \leq 3\} \subset \mathcal{H}^A \otimes \mathcal{H}^B$ can not be distinguished by PPT POVM, with $\mathcal{H}^A = \mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}$, $\mathcal{H}^B = \mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1}$ where

$$\begin{aligned}
|\chi_0\rangle_{AB} &= |\Psi_0\rangle_{A_0B_0} \otimes |\Psi_0\rangle_{A_1B_1}, \\
|\chi_1\rangle_{AB} &= |\Psi_1\rangle_{A_0B_0} \otimes |\Psi_1\rangle_{A_1B_1}, \\
|\chi_2\rangle_{AB} &= |\Psi_2\rangle_{A_0B_0} \otimes |\Psi_1\rangle_{A_1B_1}, \\
|\chi_3\rangle_{AB} &= |\Psi_3\rangle_{A_0B_0} \otimes |\Psi_1\rangle_{A_1B_1}.
\end{aligned}$$

$|\Psi_i\rangle$ are Bell states such that $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\Psi_1\rangle = (I \otimes Z)|\Psi_0\rangle$,$|\Psi_2\rangle = (I \otimes X)|\Psi_0\rangle$ and $|\Psi_3\rangle = (I \otimes Y)|\Psi_0\rangle$, where I, X, Y, Z are Pauli matrices,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Clearly, $\{|\chi_i\rangle_{AB} : 0 \leq i \leq 3\}$ can be encoded into a set of orthogonal $4 \otimes 4$ MES, as follows:

$$\begin{aligned}
|\chi_0\rangle_{AB} &= \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle), \\
|\chi_1\rangle_{AB} &= \frac{1}{2}(|00\rangle - |11\rangle - |22\rangle + |33\rangle), \\
|\chi_2\rangle_{AB} &= \frac{1}{2}(|02\rangle - |13\rangle + |20\rangle - |31\rangle), \\
|\chi_3\rangle_{AB} &= \frac{i}{2}(|02\rangle - |13\rangle - |20\rangle + |31\rangle).
\end{aligned}$$

Here we adopt the notation that $|jk\rangle = |j\rangle_A|k\rangle_B$ and $|0\rangle_A = |0\rangle_{A_0}|0\rangle_{A_1}$, $|1\rangle_A = |0\rangle_{A_0}|1\rangle_{A_1}$, $|2\rangle_A = |1\rangle_{A_0}|0\rangle_{A_1}$, $|3\rangle_A = |1\rangle_{A_0}|1\rangle_{A_1}$, and likewise for B's part.

Let us denote $C = \{I \otimes I, X \otimes X, Y \otimes Y, Z \otimes Z\}$ and $\Psi_i = |\Psi_i\rangle\langle\Psi_i|$ for $0 \leq i \leq 3$. The following two propositions will be used in the rest of this paper. The first proposition gives an explicit expression of the partial transform of $2 \otimes 2$ mixed states that are diagonal under Bell states, thus completely characterizes its separability. These two proposition can be proven directly by some routine calculation.

Proposition 1. *Let* $\rho = \sum_{i=0}^{3} \nu_i \Psi_i$ *be mixed state that are diagonal under Bell states. Then* $\rho^\Gamma = \sum_{i=0}^{3} \mu_i \Psi_i$ *with* $\mu_i = \mathrm{Tr}\rho/2 - \nu_{4-i}$. *So we have* ρ *is separable iff* $\rho, \rho^\Gamma \geq 0$ *iff* $0 \leq 2\nu_i \leq \mathrm{Tr}\rho$ *for* $0 \leq i \leq 3$.

Proposition 2. *For any linear operator* M *acting on* $2 \otimes 2$ *system,* $\sum_{\sigma \in C} \sigma M \sigma^\dagger$ *is diagonalizable under Bell states.*

Now we state our main theorem in the following:

Theorem 2. *S can not be distinguished under PPT POVM.*

The technical proof of this theorem is postponed to Appendix A.

Since LOCC is a subset of PPT operations, one can conclude that S is locally indistinguishable. Interestingly, with a $2 \otimes 2$ MES $|\Psi_0\rangle$ sharing between systems $\mathcal{H}^A$ and $\mathcal{H}^B$ as resource, there is an LOCC protocol which can distinguish S and leave us with another $|\Psi_0\rangle$: Firstly, observe that the state of subsystem $\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0}$ is an element of $\{|\Psi_i\rangle : 0 \leq i \leq 3\}$. One can distinguish them by using $|\Psi_0\rangle$. If the discrimination output is i, then one would know that the original state is $|\chi_i\rangle$. After that, if the result is 0, then the state of subsystem $\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}$ is $|\Psi_0\rangle$. Otherwise, the state of subsystem $\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}$ is $|\Psi_1\rangle$, then one will get $|\Psi_0\rangle$ by applying Z on system $\mathcal{H}^{B_1}$. Here, state $|\Psi_0\rangle$ acts like catalysts in a chemical reaction: it helps to implement otherwise impossible local distinguishing operations, but without being consumed by the operations themselves. This novel phenomenon is called "Entanglement Catalysis Discrimination". It is like the surprising phenomena "entanglement catalysis" discovered by Jonathan and Plenio [25].

IV. TENSOR PRODUCT CAN INCREASE LOCAL DISTINGUISHABILITY

In Ref. [5], Walgate et al showed that for any set of multipartite orthogonal pure states $A = \{|a_i\rangle : 0 \leq i \leq n-1\}$, $A^{n-1} = \{|a_i\rangle^{\otimes n-1} : 0 \leq i \leq n-1\}$ is always locally distinguishable. A really interesting problem is to consider the local distinguishability of $A^{\otimes k}$, where the tensor product $S_1 \otimes S_2$ of two set S_1, S_2 is given by $\{|s_1\rangle \otimes |s_2\rangle : |s_i\rangle \in S_i\}$. Note that if A is locally distinguishable, then $A^{\otimes k}$ can also be distinguished locally. On the other hand, for locally indistinguishable A, it seems that $A^{\otimes k}$ could be very difficult to distinguish since there are n^k states in $A^{\otimes k}$. This intuition is actually not true. We will next show that there exists a set of states K with the following property: K is PPT indistinguishable, but $K^{\otimes m}$ becomes locally distinguishable for some m.

Intuitively, if n states can not be distinguished by PPT POVM, then by sharing a sufficiently small entanglement, they still can not be distinguished since the set of PPT POVMs $(M_i)_{i=0}^{n-1}$ is closed. The following lemma gives a non-trivial lower bound of the entanglement cost for PPT state discrimination.

Lemma 1. *Suppose the optimal discrimination probability of bipartite orthogonal states* $\{|\varphi_i\rangle : 0 \leq i \leq n-1\}$ *on* $\mathcal{H}^A \otimes \mathcal{H}^B$ *with a priori probability distribution* $\{p_0, \cdots, p_{n-1}\}$ *by PPT POVM is* $q < 1$. *Then* $\{|\varphi_i\rangle \otimes |\alpha\rangle : 0 \leq i \leq n-1\}$ *is still PPT indistinguishable where* $|\alpha\rangle_{AB} = \sqrt{1-\varepsilon}|00\rangle + \sqrt{\varepsilon}|11\rangle$ *with* $0 \leq \varepsilon < (1-q)^2$.

Proof:—Let $|\alpha\rangle\langle\alpha| - |00\rangle\langle 00| = Q - S$, where Q and S are rank-1 positive operators with orthogonal support and $\mathrm{Tr}Q = \mathrm{Tr}S$. Thus we have $\mathrm{Tr}((|\alpha\rangle\langle\alpha| - |00\rangle\langle 00|)^2) = \mathrm{Tr}(Q^2 + S^2) = 2(\mathrm{Tr}Q)^2$, where we have employed the fact that $\mathrm{tr}Q^2 = (\mathrm{tr}Q)^2$ and $\mathrm{tr}S^2 = (\mathrm{tr}S)^2$ as both Q and S are rank-one positive operators. So $\mathrm{Tr}Q = \sqrt{\varepsilon}$.

Clearly, it holds that for any PPT POVM $(E_i)_{i=0}^{n-1}$,

$$\sum_{i=0}^{n-1} p_i \mathrm{Tr}(E_i |\varphi_i\rangle\langle\varphi_i|) \leq q.$$

For any PPT POVM $(F_i)_{i=0}^{n-1}$, the success discrimination probability of $\{|\varphi_i\rangle \otimes |\alpha\rangle : 0 \leq i \leq n\}$ with a prior probability

distribution $\{p_0,\cdots,p_{n-1}\}$ is

$$\begin{aligned}
& \sum_i p_i \mathrm{Tr}(F_i(|\varphi_i\rangle\langle\varphi_i|\otimes|\alpha\rangle\langle\alpha|)) \\
= & \sum_i (p_i \mathrm{Tr}(F_i(|\varphi_i\rangle\langle\varphi_i|\otimes|00\rangle\langle 00|)) + p_i \mathrm{Tr}(F_i(|\varphi_i\rangle\langle\varphi_i|\otimes(Q-S)))) \\
\leq & \ q + \sum_i p_i \mathrm{Tr}(F_i(|\varphi_i\rangle\langle\varphi_i|\otimes Q)) \\
\leq & \ q + \sum_i p_i \mathrm{Tr}(|\varphi_i\rangle\langle\varphi_i|\otimes Q) \\
= & \ q + \mathrm{Tr}Q \\
= & \ q + \sqrt{\varepsilon}.
\end{aligned}$$

Note that $\varepsilon < (1-q)^2$, one can therefore conclude $\{\varphi_i\otimes|\alpha\rangle : 0\leq i\leq n-1\}$ can not be distinguished by PPT POVM. ■

Theorem 2 and Lemma 1 enable us to choose a partially entangled state $|\beta\rangle_{AB} = \sqrt{1-\delta}|00\rangle + \sqrt{\delta}|11\rangle$ with $0<\delta<1/2$, such that $K = S\otimes|\beta\rangle$ is PPT indistinguishable.

Then we are able to show that tensor product can release nonlocality:

Theorem 3. *There is some m, such that $K^{\otimes m}$ can be distinguished under LOCC.*

Proof:—There exists N such that one can transform $|\beta\rangle^{\otimes N}$ to two-qubit MES $|\Psi_0\rangle$ perfectly by LOCC. $N_\delta = \lceil -\frac{1}{\log(1-\delta)}\rceil$ satisfies the conditions for entanglement transformation in Ref. [26].

Now assume $m = N_\delta$, and we shall prove $K^{\otimes m}$ is locally distinguishable. Actually $K^{\otimes m} = S^{\otimes m}\otimes|\beta\rangle^{\otimes m}$ can be distinguished by the following two steps LOCC protocol:

Step 1: Transform $|\beta\rangle_{AB}^{\otimes m}$ into $|\Psi_0\rangle$ by LOCC.

Step 2: Use $|\Psi_0\rangle$ to distinguish $S^{\otimes m}$: For any state $|\chi_{i_1}\rangle\otimes|\chi_{i_2}\rangle\otimes\cdots\otimes|\chi_{i_m}\rangle\in S$, by using $|\Psi_0\rangle$, we can identify i_1 and get another $|\Psi_0\rangle$, then identify i_2 and obtain $|\Psi_0\rangle$ again, etc. After identifying $i_1, i_2\cdots i_m$, the discrimination is finished. ■

LOCC discrimination has important implications in channel capacity [6, 22]. In Ref. [22], the authors showed that there exists a quantum channel with one sender and two receivers, of which a single use is not able to transmit information yet two uses can. For quantum channel with one sender and one receivers, recently, it was shown that entangled inputs cannot make imperfect quantum channels perfect by Brandao et.al. More precisely, for any such quantum channel, multi-use can never render noisy quantum channels have maximum capacity [23], even asymptotically.

Applying our result, we can construct a channel $\mathcal{E}$ with one classical sender Alice and two quantum receivers Bob and Charlie as follows: For input information $0\leq i\leq 3$, the output bipartite state $|\chi_i\rangle_{BC}\otimes|\beta\rangle_{BC}$ is distributed between Bob and Charlie. According to Theorem 2 and lemma 1, Bob and Charlie are not able to distinguish them perfectly, then one would know that the one-shot zero-error (or classical) capacity of $\mathcal{E}$ is strictly less than 2-bit. Suppose Alice is going to send $i_1 i_2\cdots i_m$, where m is chosen as shown in Theorem 3. Now, we know that Bob and Charlie can identify $i_1 i_2\cdots i_m$ perfectly. Thus Alice can transfer $\log_2 4^m = 2m$-bit perfectly, which means that multi-use of $\mathcal{E}$ can render this noisy quantum channels have optimal capacity. It is also worthy to note that in Ref.[22], quantum entanglement is required, but here we do not employ entangled input since the input is classical.

V. ENTANGLEMENT COST OF STATE DISCRIMINATION

In Ref. [3] it was shown that $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\}$ can not be distinguished by LOCC. It was also known that they are PPT indistinguishable. The next question will be how much entanglement is needed to distinguish them. By applying the techniques used in the proof of Theorem 2, we show that

Theorem 4. *$T = \{|\Psi_1\rangle_{A_0B_0}, |\Psi_2\rangle_{A_0B_0}, |\Psi_3\rangle_{A_0B_0}\}\otimes|\alpha\rangle_{A_1B_1}$ can be distinguished by PPT POVM iff $\lambda_0\leq 2/3$, where $|\alpha\rangle = \sum\limits_{i=0}^{n-1}\sqrt{\lambda_i}|ii\rangle$ with $\lambda_0\geq\lambda_1\geq\cdots\geq\lambda_{n-1}\geq 0$ and $\sum_{i=0}^{n-1}\lambda_i = 1$. In particular, we can choose $|\alpha\rangle_{A_1B_1} = \sqrt{2/3}|00\rangle + \sqrt{1/3}|11\rangle$.*

Later, using the same method, we obtain the following result:

Theorem 5. *If $|\Psi_0\rangle \otimes |\alpha\rangle$ and $(\Psi_1 + \Psi_2 + \Psi_3)/3 \otimes |\alpha\rangle\langle\alpha|$ can be distinguished by PPT POVM where $|\alpha\rangle = \sin\phi|00\rangle + \cos\phi|11\rangle$ with $0 \le \phi \le \pi/4$, then $|\alpha\rangle = |\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.*

Proofs of Theorem 4 and Theorem 5 are postponed to Appendices B and C, respectively.

As direct consequences of Theorem 5, we have the following interesting corollaries:

Corollary 1. *Among all $2 \otimes 2$ states, only an MES can help to distinguish between $|\Psi_1\rangle$ and $(\Psi_2 + \Psi_3 + \Psi_4)/3$ under LOCC (Separable, PPT) operations.*

Corollary 2. *Among $2 \otimes 2$ states, only MES can help to distinguish a two-qubit basis $\{|\varphi_0\rangle, |\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\}$ with $|\varphi_0\rangle = |\Psi_0\rangle$ under LOCC(Separable, PPT) operations.*

VI. CONCLUSIONS

We study the perfect state discrimination by PPT POVM. Firstly, we derive an upper bound of the number of $d \otimes d$ PPT distinguishable MES. Secondly, we present four orthogonal $4 \otimes 4$ MES which can not be distinguished by PPT POVM. Later, the Phenomena, "Entanglement Catalyst Discrimination", is observed. Based on this result, we show that there is a set K which is locally indistinguishable, but $K^{\otimes m}$ ia able to be distinguished by LOCC for some m. Then we construct a $c - q$ noisy channel where multi-use can make it perfect. Finally, this technique is used to deal with the problem of entanglement cost for state discrimination under PPT POVM.

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Appendix A: Proof of Theorem 2

Let Q denote the set of PPT POVMs that can distinguish S, i.e.

$$Q = \{(M_i)_{i=0}^3 : M_i|\chi_i\rangle = |\chi_i\rangle, \sum_i M_i = I_{16}, M_i, M_i^\Gamma \ge 0\}.$$

Clearly Q is a convex compact set. We only need to show that $Q = \emptyset$.

By Contradiction, and assume Q is non-empty. We shall first show that there exists matrices $M^{(ij)}$ acting on system $\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}$ such that $(M_i)_{i=0}^3 \in Q$, where

$$M_0 = \Psi_0 \otimes M^{(00)} + \Psi_1 \otimes M^{(01)} + \Psi_2 \otimes M^{(01)} + \Psi_3 \otimes M^{(01)}, \quad (1)$$

$$M_1 = \Psi_0 \otimes M^{(10)} + \Psi_1 \otimes M^{(11)} + \Psi_2 \otimes M^{(12)} + \Psi_3 \otimes M^{(12)}, \quad (2)$$

$$M_2 = \Psi_0 \otimes M^{(10)} + \Psi_1 \otimes M^{(12)} + \Psi_2 \otimes M^{(11)} + \Psi_3 \otimes M^{(12)}, \quad (3)$$

$$M_3 = \Psi_0 \otimes M^{(10)} + \Psi_1 \otimes M^{(12)} + \Psi_2 \otimes M^{(12)} + \Psi_3 \otimes M^{(11)}. \quad (4)$$

To see the existence of such $(M_i)_{i=0}^3$, we choose $(D_i)_{i=0}^3 \in Q$. Then for any $\sigma \in C$, we will have $(D'_i)_{i=0}^3 \in Q$ where $D'_i = \sigma_{A_0B_0} D_i \sigma^\dagger{}_{A_0B_0}$. Thus, $(E_i)_{i=0}^3 \in Q$, where

$$E_i = \frac{1}{4} \sum_{\sigma \in C} \sigma_{A_0B_0} D_i \sigma^\dagger{}_{A_0B_0},$$

for $0 \le i \le 3$. Invoking proposition 2, one would know that there exist $E^{(ij)}$ such that $E_i = \sum_{j=0}^3 \Psi_j \otimes E^{(ij)}$.

Let $W = w^* \otimes w$ with $w = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$. It is clear that $W|\Psi_0\rangle = |\Psi_0\rangle$, $W|\Psi_1\rangle = |\Psi_2\rangle$, $W|\Psi_2\rangle = |\Psi_3\rangle$ and $W|\Psi_3\rangle = |\Psi_1\rangle$. Then one may have

$$(F_i)_{i=0}^3 = W^\dagger_{A_0B_0}(E_0, E_2, E_3, E_1) W_{A_0B_0} \in Q,$$
$$(G_i)_{i=0}^3 = W_{A_0B_0}(E_0, E_3, E_1, E_2) W^\dagger_{A_0B_0} \in Q,$$

where the notation $(S_i)_{i=0}^{k-1} = W(J_i)_{i=0}^{k-1} W^\dagger$ describes the relation that $S_i = W J_i W^\dagger$ holds for any $0 \le i \le k-1$.

Now one may know that

$$(K_i)_{i=0}^3 = \left(\frac{E_i + F_i + G_i}{3}\right)_{i=0}^3 \in Q.$$

Note that $W^3 = I$, one can therefore conclude that $K_3 = W_{A_0B_0}K_2W_{A_0B_0}^\dagger$ and $K_2 = W_{A_0B_0}K_1W_{A_0B_0}^\dagger$, then there exists $D^{(ij)}$ such that

$$\begin{aligned}
K_0 &= \Psi_0 \otimes K^{(00)} + \Psi_1 \otimes K^{(01)} + \Psi_2 \otimes K^{(01)} + \Psi_3 \otimes K^{(01)},\\
K_1 &= \Psi_0 \otimes K^{(10)} + \Psi_1 \otimes K^{(11)} + \Psi_2 \otimes K^{(12)} + \Psi_3 \otimes K^{(13)},\\
K_2 &= \Psi_0 \otimes K^{(10)} + \Psi_1 \otimes K^{(13)} + \Psi_2 \otimes K^{(11)} + \Psi_3 \otimes K^{(12)},\\
K_3 &= \Psi_0 \otimes K^{(10)} + \Psi_1 \otimes K^{(12)} + \Psi_2 \otimes K^{(13)} + \Psi_3 \otimes K^{(11)}.
\end{aligned}$$

The difference between $K^{(ij)}$ and the wanted $M^{(ij)}$ is that $K^{(12)}$ maybe do not equal to $K^{(13)}$, to deal with this, let $U = u^* \otimes u$ with $u = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Clearly, $U|\Psi_0\rangle = |\Psi_0\rangle$, $U|\Psi_1\rangle = |\Psi_1\rangle$, $U|\Psi_2\rangle = |\Psi_3\rangle$ and $U|\Psi_3\rangle = -|\Psi_2\rangle$. Then the following POVM $(L_i) \in Q$, where

$$\begin{aligned}
L_0 &= \Psi_0 \otimes L^{(00)} + \Psi_1 \otimes L^{(01)} + \Psi_2 \otimes L^{(01)} + \Psi_3 \otimes L^{(01)},\\
L_1 &= \Psi_0 \otimes L^{(10)} + \Psi_1 \otimes L^{(11)} + \Psi_2 \otimes L^{(13)} + \Psi_3 \otimes L^{(12)},\\
L_2 &= \Psi_0 \otimes L^{(10)} + \Psi_1 \otimes L^{(12)} + \Psi_2 \otimes L^{(11)} + \Psi_3 \otimes L^{(13)},\\
L_3 &= \Psi_0 \otimes L^{(10)} + \Psi_1 \otimes L^{(13)} + \Psi_2 \otimes L^{(12)} + \Psi_3 \otimes L^{(11)}.
\end{aligned}$$

To see it is a PPT POVM, one only need to verify that $L_1 = U_{A_0B_0}K_1U_{A_0B_0}^\dagger$, $L_2 = W_{A_0B_0}L_1W_{A_0B_0}^\dagger$ and $L_3 = W_{A_0B_0}L_2W_{A_0B_0}^\dagger$.

Now we can obtain $(M_i)_{i=0}^3$ satisfies Eq.(1,2,3,4):

$$(M_i)_{i=0}^3 = \left(\frac{K_i + L_i}{2}\right)_{i=0}^3 \in Q.$$

It is clear that $(M_i^*)_{i=0}^3 \in Q$, where M_i^* denotes the complex conjugate of M_i. Also, for any diagonal unitary $v_{A_1} = v_{B_1}^*$ or $v_{A_1} = v_{B_1}^* = X$, $(M_i'')_{i=0}^3 \in Q$ with $M_i'' = (v_{A_1} \otimes v_{B_1})M_i(v_{A_1} \otimes v_{B_1})^\dagger$.

Now we can find that $(N_i)_{i=0}^3 \in Q$, where

$$N_i = \frac{1}{2}(\int_{V_{A_1B1}} (V_{A_1B1}M_iV_{A_1B_1}^\dagger + V_{A_1B1}M_i^*V_{A_1B_1}^\dagger)dV_{A_1B1},$$

where V_{A_1B1} ranges over all unitaries with form $v_{A_1} \otimes v_{B_1}$ for $v_{A_1} = v_{B_1}^*$ being diagonal unitaries or $v_{A_1} = v_{B_1}^* = X$. A routine calculation leads us to the fact that there exists P_i R_i and T_i are Bell diagonal Hermitians acting on system $\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0}$ such that

$$N_i = \begin{pmatrix} P_i & 0 & 0 & T_i \\ 0 & R_i & 0 & 0 \\ 0 & 0 & R_i & 0 \\ T_i & 0 & 0 & P_i \end{pmatrix}, \quad N_i^\Gamma = \begin{pmatrix} P_i^\Gamma & 0 & 0 & 0 \\ 0 & R_i^\Gamma & T_i^\Gamma & 0 \\ 0 & T_i^\Gamma & R_i^\Gamma & 0 \\ 0 & 0 & 0 & P_i^\Gamma \end{pmatrix}. \tag{5}$$

Also $N_1 = U_{A_0B_0}N_1U_{A_0B_0}^\dagger$, $N_2 = W_{A_0B_0}N_1W_{A_0B_0}^\dagger$ and $N_3 = W_{A_0B_0}N_2W_{A_0B_0}^\dagger$.

The rest part of the proof is to show such $(N_i)_{i=0}^3$ can not be some PPT POVM which can distinguish S, then we can conclude that Q is empty.

Since P_i R_i and T_i are Bell diagonal, thus P_i^Γ R_i^Γ and T_i^Γ are Bell diagonal. Then, we will have the following inequalities:

$$\begin{aligned}
N_i \geq 0 &\Rightarrow P_i \geq |T_i|, R_i \geq 0,\\
N_i^\Gamma \geq 0 &\Rightarrow P_i^\Gamma \geq 0,\ R_i^\Gamma \geq |T_i^\Gamma|,
\end{aligned}$$

where $|A|$ is used to denote the positive square root of $A^\dagger A$.

According to Eq.(5,1), we can assume $P_1 = a_0\Psi_0 + a_1\Psi_1 + a_2\Psi_2 + a_2\Psi_3$ for some $a_0, a_1, a_2 \geq 0$ and $T_1 = b_0\Psi_0 + b_1\Psi_1 + b_2\Psi_2 + b_2\Psi_3$, then

$$N_1|\chi_j\rangle = \delta_{1j}|\chi_i\rangle \Rightarrow a_1 - b_1 = 1, a_2 - b_2 = 0, a_0 + b_0 = 0$$

Together with the relations Eq.(1,5) and $N_0 = I_{16} - N_1 - N_2 - N_3$, one would have

$$\begin{aligned} P_1 &= a_0\Psi_0 + a_1\Psi_1 + a_2\Psi_2 + a_2\Psi_3, \\ P_0 &= (1-3a_0)\Psi_0 + (1-a_1-2a_2)(I-\Psi_0). \\ T_1 &= -a_0\Psi_0 + (a_1-1)\Psi_1 + a_2\Psi_2 + a_2\Psi_3, \\ T_0 &= 3a_0\Psi_0 + (1-a_1-2a_2)(I-\Psi_0). \end{aligned}$$

Then

$$P_1{}^\Gamma \geq 0 \Rightarrow a_1 \leq a_0 + 2a_2, \tag{6}$$

$$P_1 \geq |T_1| \Rightarrow a_i \geq 0, |a_1 - 1| \leq a_1 \Rightarrow 1/2 \leq a_1 \leq 1, \tag{7}$$

$$P_0 \geq |T_0| \Rightarrow a_1 + 2a_2 \leq 1, 3a_0 \leq (1-3a_0) \Rightarrow 0 \leq a_0 \leq 1/6. \tag{8}$$

Now it holds that

$$P_0{}^\Gamma \geq 0 \Rightarrow 1 - 3a_0 \leq 3 - 3(a_1 + 2a_2) \Rightarrow 0 \leq 1 - 6a_0 \leq 3 - 3(a_1 + 2a_2 + a_0) \leq 3 - 3(a_1 + a_1) \leq 0. \tag{9}$$

Thus

$$a_1 = 1/2, a_2 = a_0 = 1/6. \tag{10}$$

We will have that $|T_i^\Gamma| = 1/3(\Psi_0 + \Psi_1 + \Psi_2)$ is valid for any $0 \leq i \leq 3$. Then $R_i^\Gamma \geq |T_i| = 1/3(\Psi_0 + \Psi_1 + \Psi_2)$.

It is easy to obtain $I_4 = \sum\limits_{i=0}^{3} R_i$ from $\sum\limits_{i=0}^{3} N_i = I_{16}$. Then $I_4 = \sum\limits_{i=0}^{3} R_i^\Gamma \geq \sum\limits_{i=0}^{3} |T_i| = \frac{4}{3}(\Psi_0 + \Psi_1 + \Psi_2)$, which is impossible! Thus, $(N_i)_{i=0}^3$ is not any PPT POVM which can distinguish S, which means that Q is an empty set.

Therefore, $\{|\chi_i\rangle_{AB} : 0 \leq i \leq 3\}$ are PPT indistinguishable. ■

Appendix B: Proof of Theorem 4

Suppose $(M_i)_{i=1}^3$ be some PPT POVM which can distinguish T, then, PPT POVM $(E_i)_{i=1}^3$ can also distinguish T, where

$$E_i = \frac{1}{2}\left(\int\limits_V VM_iV^\dagger + \int\limits_V VM_i^*V^\dagger\right)dV,$$

where V ranges over all unitaries with form $\sigma_{A_0B_0} \otimes u_{A_1} \otimes u_{B_1}$ for $\sigma_{A_0B_0} \in C$ and diagonal unitaries $u_{A_1} = u_{B_1}^*$. Similar as the proof of Theorem 2, we can assume that

$$E_2 = W_{A_0B_0}E_1W_{A_0B_0}^\dagger, E_3 = W_{A_0B_0}E_2W_{A_0B_0}^\dagger, \tag{11}$$

where $W = w^* \otimes w$ and $w = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$.

Let $E_1 = \sum\limits_{ij}(N^{(ij)} \otimes |ii\rangle\langle jj| + R^{(ij)} \otimes |ij\rangle\langle ij|)$, where $N^{(ij)}$ and $R^{(ij)}$ are Hermitian acting on A_0B_0 with eigenvectors $|\Psi_k\rangle$. Assume $N^{(ij)} = a_{ij}\Psi_0 + b_{ij}\Psi_1 + c_{ij}\Psi_2 + d_{ij}\Psi_3$. According to $E_1 + W_{A_0B_0}E_1W_{A_0B_0}^\dagger + W_{A_0B_0}^2E_1{W^2}^\dagger_{A_0B_0} = I_{A_0B_0A_1B_1}$, one can conclude that

$$a_{00} = 1/3, b_{00} + c_{00} + d_{00} = 1. \tag{12}$$

From $E_1^\Gamma \geq 0$, we will know ${N^{(00)}}^\Gamma \geq 0$, then $b_{00} \leq 2/3$.

From $E_1|\Psi_1\rangle\otimes|\alpha\rangle = |\Psi_1\rangle\otimes|\alpha\rangle$, we will know that $|e\rangle = \sum_{i=0}^{n-1}\sqrt{\lambda_i}|i\rangle$ is an eigenvector corresponding to engenvalue 1 of positive matrix $M_A = (b_{ij})$. Let $M_A = |e\rangle\langle e| + C$ with $C \geq 0$, then $\lambda_0 \leq b_{00} \leq 2/3$.

In the following, we will show that $|\varphi\rangle = \sqrt{2/3}|00\rangle + \sqrt{1/3}|11\rangle$ is sufficient: Let $E_1 = N^{(00)}\otimes|00\rangle\langle 00| + N^{(01)}\otimes|01\rangle\langle 01| + N^{(10)}\otimes|10\rangle\langle 10| + N^{(11)}\otimes|11\rangle\langle 11| + R^{(01)}\otimes(|11\rangle\langle 00| + |00\rangle\langle 11|)$, where

$$\begin{aligned} N^{(00)} &= 1/3\Psi_0 + 2/3\Psi_1 + 1/6\Psi_2 + 1/6\Psi_3, \\ N^{(01)} &= 1/3\Psi_0 + 1/2\Psi_2 + 1/2\Psi_3, \\ N^{(10)} &= N^{(11)} = I/3, \\ R^{(01)} &= \sqrt{2}/3\Psi_1 - \sqrt{2}/6\Psi_2 - \sqrt{2}/6\Psi_3. \end{aligned}$$

Directly, $(E_1, W_{A_0B_0}E_1W_{A_0B_0}^\dagger, W_{A_0B_0}^2E_1W_{A_0B_0}^{2\dagger})$ is a PPT POVM which can distinguish T. This theorem is valid. ■

appendix C: Proof of Theorem 5

Let $\{M_0, M_1\}$ be some wanted PPT POVM, then $(M, I-M)$ is a PPT POVM which can distinguish $|\Psi_0\rangle\otimes|\alpha\rangle$ and $(\Psi_1+\Psi_2+\Psi_3)/3\otimes|\alpha\rangle\langle\alpha|$, where

$$M = \frac{1}{2}\left(\int_V VM_0V^\dagger + \int_V VM_0^*V^\dagger\right)dV,$$

where V ranges over all unitaries with form $v_{A_0}\otimes v_{B_0}\otimes u_{A_1}\otimes u_{B_1}$ for unitary $v_{A_0} = v_{B_0}^*$ and diagonal unitaries $u_{A_1} = u_{B_1}^*$. Then we can assume $M = N^{(00)}\otimes|00\rangle\langle 00| + N^{(01)}\otimes|01\rangle\langle 01| + N^{(10)}\otimes|10\rangle\langle 10| + N^{(11)}\otimes|11\rangle\langle 11| + R\otimes(|00\rangle\langle 11| + |11\rangle\langle 00|)$, where $N^{(ij)}$ and R are acting on the original system with $N^{(ij)} = a_{ij}\Psi_1 + b_{ij}(I_{AB}-\Psi_1)$, and $R = c\Psi_1 + d(I_{AB}-\Psi_1)$ with $a_i, b_i \geq 0$, and $c, d \in \mathcal{R}$.

$M|\Psi_0\rangle\otimes|\alpha\rangle = |\Psi_0\rangle\otimes|\alpha\rangle$ and $M|\Psi_1\rangle\otimes|\alpha\rangle = 0$ leads us to

$$\begin{aligned} a_{00} + c\cot\phi = 1, a_{11} + c\tan\phi = 1. \\ b_{00} + d\cot\phi = 0, b_{11} + d\tan\phi = 0. \end{aligned}$$

$M^\Gamma \geq 0$ and $I - M^\Gamma \geq 0$ implies

$$\begin{aligned} {N^{(00)}}^\Gamma \geq 0 \Rightarrow a_{00} \leq 3b_{00} \Rightarrow 1 \leq (c-3d)\cot\phi, \\ {N^{(11)}}^\Gamma \geq 0 \Rightarrow a_{11} \leq 3b_{11} \Rightarrow 1 \leq (c-3d)\tan\phi, \\ \langle\Psi_i|M^\Gamma|\Psi_i\rangle \geq 0 \Rightarrow (3d-c)^2 \leq (3b_{10}-a_{10})(3b_{11}-a_{11}), \\ \langle\Psi_i|I-M^\Gamma|\Psi_i\rangle \geq 0 \Rightarrow (3d-c)^2 \leq (2+a_{10}-3b_{10})(2+a_{11}-3b_{11}). \end{aligned}$$

Now

$$\begin{aligned} 1 \leq (c-3d)\cot\phi \times (c-3d)\tan\phi = (c-3d)^2, \\ (3d-c)^2(3d-c)^2 \leq (3b_{10}-a_{10})(3b_{11}-a_{11})(2+a_{10}-3b_{10})(2+a_{11}-3b_{11}) \leq 1. \end{aligned}$$

Thus $|c-3d| = 1$ and $|\tan\phi| = 1$. The proof of this theorem is complete. ■

[1] M. Nathanson, J. Math. Phys. **46**, 062103 (2005).
[2] M. Hayashi, D. Markham, M. Murao, M. Owari and S. Virmani, Phys. Rev. Lett. **96**, 040501 (2006).
[3] S. Ghosh, G. Kar, A. Roy, A. Sen(De), and U. Sen, Phys. Rev. Lett. **87**, 277902 (2001).
[4] S. Ghosh, G. Kar, A. Roy, and D. Sarkar, Phys. Rev. A **70**, 022304 (2004).
[5] J. Walgate, A. J. Short, L. Hardy and V. Vedral, Phys. Rev. Lett. **85**, 4972 (2000).
[6] J. Watrous, Phys. Rev. Lett. **95**, 080505 (2005).
[7] M. Horodecki, J. Oppenheim, A. Sen(De) and U. Sen, Phys. Rev. Lett. **93**, 170503 (2004).

[8] J. Walgate and L. Hardy, Phys. Rev. Lett. **89**, 147901 (2002).
[9] R.Duan, Y. Feng, Z. Ji, and M. Ying, Phys. Rev. Lett. **98**, 230502 (2007).
[10] R. Duan, Y. Feng, Y. Xin, M. Ying, IEEE Trans. Inf. Theory **55**, 1320 (2009).
[11] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Phys. Rev. A, **59**, 1070 (1999).
[12] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B.M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).
[13] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B.M. Terhal, Comm. Math. Phys. **238**, 379 (2003).
[14] Y. Feng and Y. Shi, IEEE Trans. Inf. Theory, **55**, 6 (2009).
[15] H. Fan, Phys. Rev. Lett. vol. **92**, 177905 (2004).
[16] M. Owari and M. Hayashi, Phys. Rev. A, **74**, 032108 (2006).
[17] E. M. Rains, IEEE Trans. Inf. Theory, **47**, 7 (2001).
[18] W. Matthews and A. Winter, Phys. Rev. A **78**, 012317 (2008).
[19] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[20] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[21] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 24 (1996).
[22] R. Duan and Y. Shi, Phys. Rev. Lett. **101**, 020501 (2008).
[23] F. G. S. L. Brandao, J. Eisert, M. Horodecki, and D. Yang, Phys. Rev. Lett. **106**, 230502(2011).
[24] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).
[25] D. Jonathan and M.B. Plenio, Phys. Rev. Lett. **83**, 1455 (1999).
[26] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).